Distance Coloring

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Abstract

Given a graph $G = (V, E)$, a (d, k)-coloring is an assignment of a color from $\{1, 2, \ldots, k\}$ to each vertex of V such that any two vertices within distance d of each other are assigned different colors. We determine the complexity of the (d, k)-coloring problem for all d and k, and enumerate some interesting properties of (d, k)-colorable graphs. Our main result is the discovery of a dichotomy between polynomial and NP-hard instances; for fixed $d \geq 2$, the distance coloring problem is polynomial time for $k \leq \lfloor \frac{3d}{2} \rfloor$ and NP-hard for $k > \lfloor \frac{3d}{2} \rfloor$.

1 Introduction

The classic *k-coloring problem* tries to assign a color from 1 to k to each vertex in a graph G such that any two adjacent vertices are assigned different colors [1]. The k-coloring problem, along with many variations and generalizations, has been extensively studied in both computer science and mathematics [2, 3, 4, 5, 6]. Its applications range from frequency assignment [7] and register allocation [8, 9, 10] to circuit board testing [11] and time table scheduling [12].

The *distance (d, k)-coloring problem* is a generalization of k-coloring that tries to assign a color from 1 to k to each vertex such that vertices within distance d of each other are assigned different colors. Clearly, k-coloring is a special case of (d, k)-coloring with $d = 1$. Conversely, (d, k)-coloring a graph G is equivalent to k-coloring G^d, the *dth* power graph of G. The graph G^d has the same vertex set as G and an edge between two vertices if and only if they are within distance d of each other in G. In this way (d, k)-coloring is also a special case of k-coloring.

Although k-coloring is NP-complete for $k \geq 3$ and polynomial-time for $k \leq 2$, the complexity of (d, k)-coloring is not so straightforward. In fact, there are many values of d and k for which (d,k)-coloring has a polynomial-time algorithm. The dichotomy between polynomial and NP-hard instances is the subject of this paper. The results are not only of theoretical interest, but can also be used in practice in applications with underlying structures that fit the power graph model. For example, we may want to assign k frequencies to a set of radio stations, but require that any two stations within distance d of each other use different frequencies.

The main result is quite remarkable, in that it classifies all instances of (d,k)-coloring for $d \geq 2$: determining whether a graph is (d,k)-colorable is solvable in polynomial-time for $k \leq \lfloor \frac{3d}{2} \rfloor$, but is NP-hard for $k > \lfloor \frac{3d}{2} \rfloor$. Also, (d,k)-coloring on trees is shown to be polynomial-time solvable.

Open problems remain for (d,k)-coloring, including approximation algorithms and hardness of approximation results for those parameters for which (d,k)-coloring is NP-hard. Also, exploring the relationship between the chromatic numbers of G and G^d would prove interesting.

2 Preliminaries

Given a graph $G = (V, E)$ and integers $d \geq 1$, $k \geq d+1$, the (d,k) *coloring problem* is to find an assignment of k colors to the vertices of G such that vertices within distance d of each other are assigned different colors. More precisely, the goal is to find a function $f : V \to \{1, 2, \ldots, k\}$ such that $d(u,v) \leq d \Rightarrow f(u) \neq f(v)$. For shorthand, we say G is (d,k)-*colorable* if G is distance d, k-colorable, and use (d,k)-*coloring* to mean a distance d, k-coloring. The standard k-coloring problem is the special case when $d = 1$.

Assumptions. Notice that each connected component of G can be colored independently. Moreover, any component of size less than or equal to k can be (d,k)-colored by assigning a distinct color to each vertex. Thus we assume that G is connected and has at least $k+1$ vertices.

Definition 2.1 *For a vertex v and integer $r \geq 0$, let G_v^r denote the* subgraph of radius r around G, *i.e.* $G_v^r = \{w \mid d(w,v) \leq r\}$.

Definition 2.2 *For a subgraph $G' \subseteq G$, the* diameter of G', *denoted $diam(G')$ is defined as the maximum shortest path distance between any two vertices of G', i.e., $diam(G') = \max\{d(u,v) \mid u, v \in G'\}$.*

Definition 2.3 *Given a connected graph G, a set of vertices $V' \subseteq V$ is a* cutset *if the removal of V' disconnects G into two or more non-trivial connected components.*

Definition 2.4 *Given a connected graph* G *of size at least* $k+1$*, a* forbidden (d, k) subgraph *is a subgraph* $G' = (V', E') \subseteq G$ *such that* $|V'| \leq k+1$ *and* $diam(G') \leq |V'| - (k-d) - 1$*.*

2.1 Properties of (d, k)-colorable graphs

Theorem 2.5 *If* G *contains a forbidden* (d, k) *subgraph then it is not* (d, k)*-colorable.*

Proof. Suppose G is a connected graph of size $\geq k+1$ that contains a subgraph $G' = (V', E')$ with $|V'| \leq k+1$ and $diam(G') \leq |V'| - (k-d) - 1$. Let G'' be a connected graph obtained from G' and $k+1-|V'|$ vertices of $G \backslash G'$; G'' has $k+1$ vertices and diameter at most $diam(G')+k+1-|V'| = d$. Thus G contains $k+1$ vertices all within distance d of each other, which precludes a (d, k)-coloring. $\square$

Corollary 2.6 *If there is a vertex* $v \in G$ *such that* $|G_v^r| \geq (k-d) + 2r + 1$ *for any* $2 \leq r \leq \lfloor \frac{d}{2} \rfloor$ *then* G *cannot be* (d, k)*-colored.*

Proof. The diameter of G_v^r is at most $2r \leq d$. But $2r = (k-d+2r+1) - (k-d) - 1 \leq |G_v^r| - (k-d) - 1$ and Theorem 2.5 completes the proof. $\square$

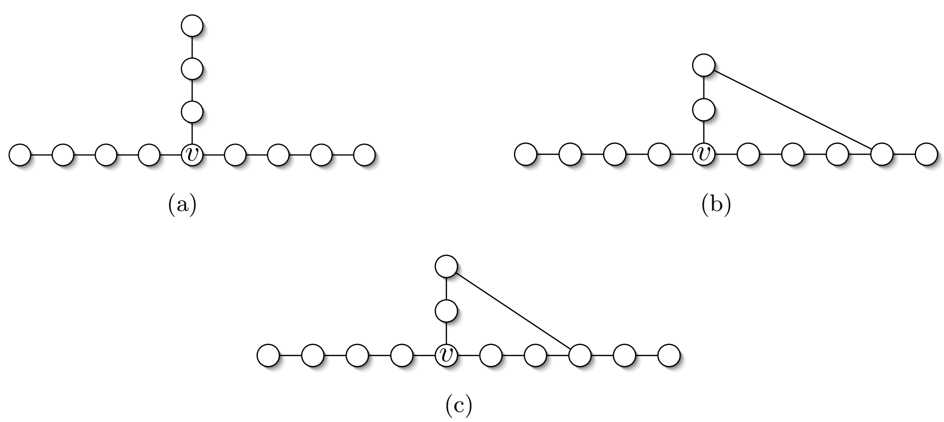

Figure 1: Examples of Corollary 2.6 for $(6, 9)$-coloring. 1(a), 1(b) are such that $|G_v^3| = 10$ and G cannot be $(6, 9)$-colored. 1(c) is such that $|G_v^r| \leq 2r + 3 + 1$ for all $r \leq 3$ and G can be $(6, 9)$-colored.

Theorem 2.7 *Given a (d,k)-colorable graph G with $k \le \lfloor \frac{3d}{2} \rfloor$, if G_v^{k-d} is a strict subgraph of G for some $v \in G$ then G_v^{k-d} contains a cutset of size ≤ 2 disconnecting v from $G \setminus G_v^{k-d}$.*

Proof. First observe that any vertex $y \in G \setminus G_v^{k-d}$ is connected to v through at least one vertex of each $G_v^i \setminus G_v^{i-1}$ for $2 \le i \le k-d$. We will show that $\exists i \le k-d$ such that $|G_v^i \setminus G_v^{i-1}| \le 2$. Then the removal of $G_v^i \setminus G_v^{i-1}$, of at most two vertices, would disconnect y from v.

By way of contradiction, suppose that for all $i \le k-d$, $|G_v^i \setminus G_v^{i-1}| \ge 3$. Then $|G_v^0| = 1$ implies $|G_v^i| \ge 3i+1$. Hence $|G_v^{k-d}| \ge 2(k-d)+(k-d)+1$ and G is not (d,k)-colorable by Corollary 2.6, a contradiction. □

Lemma 2.8 *Given a (d,k)-colorable graph G with $k \le \lfloor \frac{3d}{2} \rfloor$ and a path P of length $0 \le p \le \lfloor \frac{d}{2} \rfloor$, let v_L and v_R be its left- and right-most endpoints, respectively. Suppose there exist disjoint subgraphs P_L, P_R in $G \setminus P$ such $P_L \cup \{v_L\}$, $P_R \cup \{v_R\}$ are connected and $|P_L|, |P_R| \ge \lfloor \frac{d}{2} \rfloor$. Let O be the non-P vertices connected to P in the graph $G \setminus (P_L \cup P_R)$. Then $|O| \le k-d-1$.*

Proof. Label P as $v_L = v_0, v_1, \ldots, v_p = v_R$. Let $O_0, O_1, \ldots, O_p$ be a partition of O such that the induced subgraph $O_i \cup \{v_i\}$ is connected for all $0 \le i \le p$. See Figure 2.

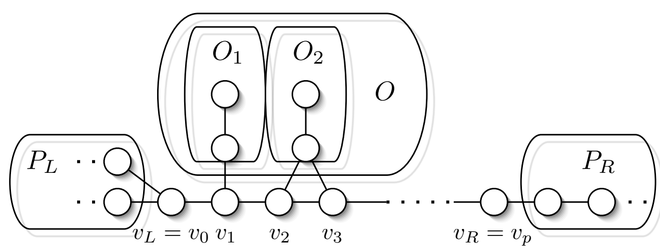

Figure 2: Notation for Lemma 2.8.

Note that $|O_i| \le k-d-1 \le \lfloor \frac{d}{2} \rfloor - 1$ for all $i \ge 0$ otherwise $G_{v_i}^{\lfloor \frac{d}{2} \rfloor}$ would be a forbidden subgraph and G not (d,k)-colorable by Corollary 2.6. Let

$$d_L = \max\{2\lfloor \frac{d}{2} \rfloor - \max_{i \ge 0}\{|O_i| + i, \lfloor \frac{d}{2} \rfloor\}, \max_{i \ge 0}\{|O_i| - i\}\} \ge \lfloor \frac{d}{2} \rfloor - p$$

Consider the subgraph G' consisting of P, the d_L closest vertices of P_L to v_L and the $2\lfloor \frac{d}{2} \rfloor - p - d_L$ closest vertices of P_R to v_R; G' consists of $2\lfloor \frac{d}{2} \rfloor + 1$

vertices within distance $2\lfloor \frac{d}{2} \rfloor$ of each other. The vertices of O are within $d_L + \max_{j\geq 0}\{|O_j|+j\}$ of the $G' \cap P_L$ vertices, where

$$\begin{aligned}
d_L + \max_{j\geq 0}\{|O_j|+j\} &= \max\{2\lfloor \frac{d}{2} \rfloor - \max_{i\geq 0}\{|O_i|+i, \lfloor \frac{d}{2} \rfloor\} + \max_{j\geq 0}\{|O_j|+j\}, \\
&\qquad \max_{i\geq 0}\{|O_i|-i\} + \max_{j\geq 0}\{|O_j|+j\}\} \\
&\leq \max\{2\lfloor \frac{d}{2} \rfloor, \max_{i\geq 0, j\neq i}\{2|O_i|, |O_i|+|O_j|+p\}\} \\
&\leq \max\{2\lfloor \frac{d}{2} \rfloor, |O|+p\}.
\end{aligned}$$

Similarly, the vertices of O are within $2\lfloor \frac{d}{2} \rfloor - p - d_L + \max_{j\geq 0}\{|O_j|+p-j\}$ of the $G' \cap P_R$ vertices, where

$$\begin{aligned}
2\lfloor \frac{d}{2} \rfloor - d_L + \max_{j\geq 0}\{|O_j|-j\} &= 2\lfloor \frac{d}{2} \rfloor - \max\{2\lfloor \frac{d}{2} \rfloor - \max_{i\geq 0}\{|O_i|+i, \lfloor \frac{d}{2} \rfloor\} \\
&\qquad - \max_{j\geq 0}\{|O_j|-j\}, 0\} \\
&\leq 2\lfloor \frac{d}{2} \rfloor.
\end{aligned}$$

Lastly, the vertices of O are at most distance $p+|O|$ from each other. Thus we have $2\lfloor \frac{d}{2} \rfloor + 1 + |O|$ vertices of distance $\leq \max\{2\lfloor \frac{d}{2} \rfloor, |O|+p\}$ from each other, and hence $|O| \leq k-d-1$ otherwise there is a forbidden subgraph. □

3 Hardness Results

Theorem 3.1 *The (d,k)-coloring problem is NP-hard for $d \geq 2$, $k > \lfloor \frac{3d}{2} \rfloor$.*

Theorem 3.1 follows via a reduction from $(1,k)$-coloring. Given a graph G that we wish to k-color, we construct a graph G' such that G is k-colorable if and only if G' is (d,k)-colorable.

Triangle Gadgets. Depending on whether d is odd or even, the triangle gadget G^Δ is shown in Figure 3.

Lemma 3.2 *If G^Δ is (d,k)-colorable then x, y and z have the same color.*

Proof. The $k-1$ vertices in $G^\Delta \setminus \{x,y,z\}$ are within distance d of each other, and use $k-1$ distinct colors. The vertices x, y and z are within

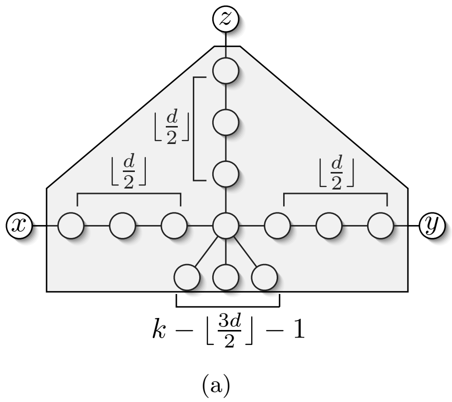

Figure 3: Gadget G^Δ used in Theorem 3.1 for odd and even d are shown in figures 3(a) and 3(b), respectively.

distance d of all these $k-1$ colors, but distance $d+1$ from each other. Thus if only k colors are used in G^Δ then x, y and z must be colored the same. □

Reduction from k-COL. Given a graph $G = (V, E)$ that we wish to k-color, create the graph $G' = (V', E')$ as follows. For each vertex $u \in G$ create gadget $G^u \in G'$ by concatenating $deg(u) \cdot 2k^4$ copies of G^Δ together, overlapping the x and y vertices, always leaving the z vertices open. Every $2k^4 th$ z vertex is reserved for use as follows: for each edge $e = (u, v) \in G$, create an edge $(u_e, v_e) \in G'$ where u_e and v_e are reserved vertices of G^u and G^v, respectively. An example of this reduction is shown in Figure 4. Note that G' is polysize, as $|G^\Delta| = k + 2$ and it is copied $deg(u) \cdot 2k^4$ times.

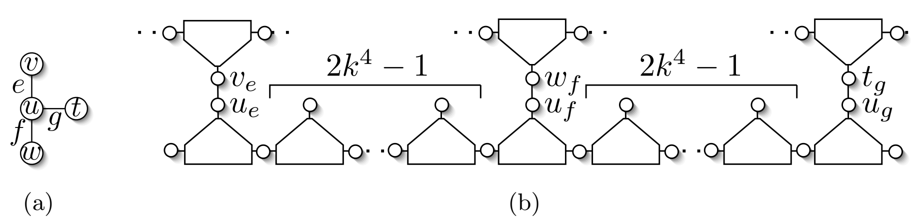

Figure 4: 4(a) An example graph G. 4(b) A subgraph of the graph G' constructed from G in Theorem 3.1.

Lemma 3.3 *If G' is (d,k)-colorable then $u_e, u_f \in G'$ have the same color for all edges e, f incident to $u \in G$.*

Proof. By construction and Lemma 3.2 we know that all x, y and z vertices in G^u must have the same color. The vertices u_e and u_f are simply z vertices, and the result follows. $\square$

Lemma 3.4 *If G' is (d,k)-colorable then for edge $e = (u,v) \in G$, $u_e, v_e \in G'$ are different colors.*

Proof. If $e = (u,v) \in G$ then $(u_e, v_e) \in G'$. Thus $d(u_e, v_e) \leq d$ and u_e, v_e are different colors in G'. $\square$

Lemma 3.5 *If G' is (d,k)-colorable then G is k-colorable.*

Proof. If G' has a (d,k)-coloring C then create a feasible k-coloring D of G by setting $D(u) = C(u_e)$ for any e incident to u. Since $C(u_e) = C(u_f)$ for all e, f incident to u by Lemma 3.3, D is well-defined. Moreover, $D(u) \neq D(v)$ for $(u,v) \in G$ by Lemma 3.4, and D uses at most k colors. $\square$

Lemma 3.6 *If G is k-colorable then G' is (d,k)-colorable.*

The proof of Lemma 3.6 requires some additional framework. Consider the colors of the vertices of $G^\Delta \setminus \{x,y,z\}$, labeled as in figure 5(a). If we use the coloring of this G^Δ to color an adjacent G^Δ as shown in figures 5(b) or 5(c) then we will have defined a permutation group based on the colors of $G^\Delta \setminus \{x,y,z\}$. The base set is $A = \{\text{colors of } G^\Delta \setminus \{x,y,z\}\}$ and we have two elements $\sigma, \pi : A \to A$ such that σ is the shift operator $(a_1\ a_2\ \cdots\ a_{k-1})$ and τ is the adjacent transposition operator $(a_1\ a_2)(a_3)(a_4)\cdots(a_{k-1})$.

Lemma 3.7 *Applying either the shift σ or the transposition τ to the colors of G^Δ yields a valid coloring.*

Proof. The vertices of the same color in adjacent G^Δs are at least distance $d+1$ apart. $\square$

Lemma 3.8 *Any adjacent transposition $\pi = (a_j\ a_{j+1})$ can be generated from k compositions of σ and τ.*

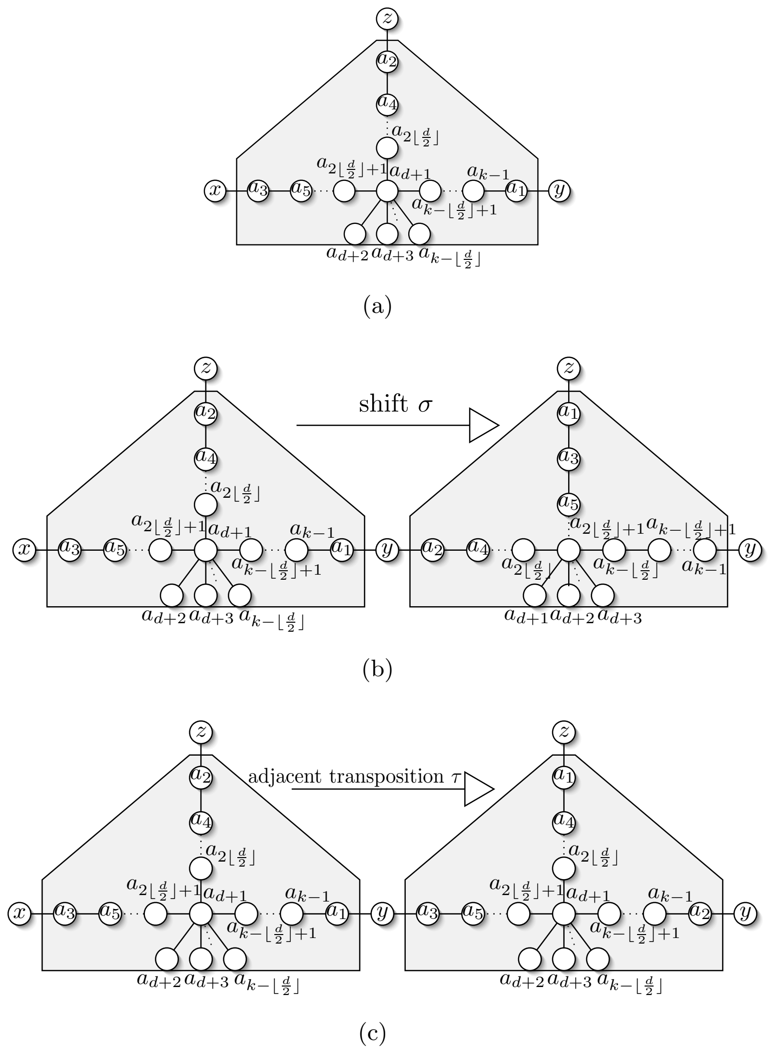

Figure 5: 5(a) The set A for the basis of the permutation group for odd d. The same labeling can be used for even d, ignoring the center vertex. 5(b) The shift operator σ. 5(c) The adjacent transposition operator τ.

Proof. Use the shift σ $j-1$ times, transpose with τ once, then shift again $(k-1)-(j-1)$ times. In this way the only elements transposed are j and $j+1$, and the remaining elements are shifted $k-1$ times and hence remain unchanged. $\square$

Lemma 3.9 *Any transposition $\pi = (a_i \; a_j)$ can be generated from $\leq 2k$ compositions of adjacent transpositions.*

Proof. Without loss of generality, suppose $i < j$. Note that

$$(a_i \, a_j) = (a_{j-1} \, a_j)(a_{j-2} \, a_{j-1}) \cdots (a_i \, a_{i+1})(a_{i+1} \, a_{i+2}) \cdots (a_{j-2} \, a_{j-1})(a_{j-1} \, a_j).$$

This uses $2(j-i)-1 \leq 2k$ transpositions, as required. □

Lemma 3.10 *Any cycle $\pi = (\pi_1 \; \pi_2 \; \cdots \; \pi_p)$ can be generated from $\leq p$ compositions of transpositions.*

Proof. Note that $(\pi_1 \; \pi_2 \; \cdots \; \pi_p) = (\pi_1 \; \pi_p)(\pi_1 \; \pi_{p-1}) \cdots (\pi_1 \; \pi_3)(\pi_1 \; \pi_2)$. □

Lemma 3.11 *Any permutation on k elements can be generated from $\leq 2k^4$ compositions of the shift σ and the adjacent transposition τ.*

Proof. First note that any permutation on A can be generated by at most k cycles of length k. By Lemmas 3.8-3.10 we have that

$$\begin{aligned} k \text{ cycles of length } k \quad &\leq \quad k^2 \text{ transpositions} \\ &\leq \quad 2k^3 \text{ adjacent transpositions} \\ &\leq \quad 2k^4 \text{ compositions of } \sigma \text{ and } \tau. \end{aligned}$$

Thus any permutation is generated by $\leq 2k^4$ compositions of σ and τ. □

Proof of Lemma 3.6. Given a k-coloring D of G we show how to construct a (d,k)-coloring of the gadgets of G'. First, for each vertex $u \in G$, color all of G^u's x, y and z vertices with the color $D(u)$. This is feasible because all these vertices are distance $d+1$ apart, and their color is different from all neighboring gadgets' x, y, z colors. Next, for each vertex $u \in G$, color the G^Δ's directly connected to G^v for some other $v \in G$. Now the remaining uncolored G^Δ gadgets in G^u are chains of G^Δs of length $2k^4$ between two colored G^Δs. We know there is some chain of at most $2k^4$ compositions of σ and τ that lead from any one coloring of G^Δ to the next, and each composition corresponds to a valid coloring of each G^Δ. Thus we know there is a sequence of colorings getting us from one G^Δ to the one of distance $2k^4$ away, and G' can be colored. □

Proof of Theorem 3.1. The polynomial reduction constructs a graph G' that is (d,k)-colorable if and only if G is k-colorable, by Lemmas 3.5 and 3.6. Therefore (d,k)-coloring is NP-complete for $d \geq 2$ and $k \geq \lfloor \frac{3d}{2} \rfloor + 1$. □

4 Algorithms

4.1 $(d, d+1)$-coloring

Theorem 4.1 *The $(d, d+1)$-coloring problem is polynomial-time solvable.*

Proof. A graph is $(1, 2)$-colorable if and only if it is bipartite.

For $d \geq 2$, Corollary 2.6 implies that a graph G is not $(d, d+1)$-colorable if it contains a vertex of degree 3 or greater. Otherwise G is a path or cycle. If it is a cycle but its length is not a multiple of $(d+1)$ then it is not $(d, d+1)$-colorable. Otherwise G is a path or a cycle whose length is a multiple of $(d+1)$. In either case, cycle through the colors $1, 2, \ldots, (d+1)$, which ensures that vertices within distance d of each other are colored differently. $\square$

4.2 (d, k)-coloring for $k \leq \lfloor \frac{3d}{2} \rfloor$

We describe an algorithm that is polynomial in $|G|$ and either gives a (d, k)-coloring of G or declares that no such coloring exists. The algorithm finds a bounded tree-width tree decomposition of G [16], then colors the graph using known coloring algorithms for graphs of bounded tree-width [17, 18].

4.2.1 Bounded Tree-width

The concepts of tree decomposition and bounded tree width were introduced in [16]. A tree decomposition of G is a triple (T, F, X) consisting of an undirected tree (T, F) and a map $X : T \rightarrow 2^V$ associating a subset $X_i \subseteq V$ with each $i \in T$ such that

(i) $V = \bigcup_{i \in T} X_i$;

(ii) for all edges $(s, t) \in E$, there exists $i \in T$ such that $(s, t) \in X_i$;

(iii) if j lies on the path between i and k in (T, F), then $X_i \cap X_k \subseteq X_j$.

The *width* of (T, F, X) is $\max_i\{|X_i| - 1\}$.

A *path decomposition* is just a tree decomposition in which the graph (T, F) is a simple path.

4.2.2 Computing a Path Decomposition

Let P be a simple path of of length $\geq d+1$ in G, and let s be the center vertex. If no such path exists then $diam(G) \leq d$ and G is not (d, k)-colorable. Otherwise, perform a breadth-first search on G starting from s to get level sets $L_0, L_1, \ldots, L_m$, where L_i consists of the set of vertices of distance i from the root s. The *level graph* H is the graph consisting of vertices V and *directed edges* from L_i to L_{i+1}, ignoring edges between vertices on the same level. We take $L_j = \emptyset$ for $j > m$. For $0 \le i \le \max\{0, m-d\}$, let

$$X_i \stackrel{def}{=} \bigcup_{j=i}^{i+d} L_j$$

Lemma 4.2 *If G is (d,k)-colorable, then $|X_i| \le 5d$.*

Proof. Let the right side of P be labeled $s = s_0^R, s_1^R, \ldots, s_{m_R}^R$ and the left side of P be labeled $s = s_0^L, s_1^L, \ldots, s_{m_L}^L$ such that $s_i^R, s_i^L \in L_i$ (where $m_L, m_R \ge \lceil \frac{d}{2} \rceil$ because $\ell(P) \ge d+1$). For $0 \le i \le m$, let T_i^R, T_i^L be the subgraphs of the level graph rooted at s_i^R, s_i^L, respectively, consisting of all vertices reachable from s_i^R, s_i^L by a (directed) path in H. Let $T^L = H \setminus T_1^R$, $T^R = H \setminus T_1^L$. Note that $H = T^L \cup T^R$. See Figure 6.

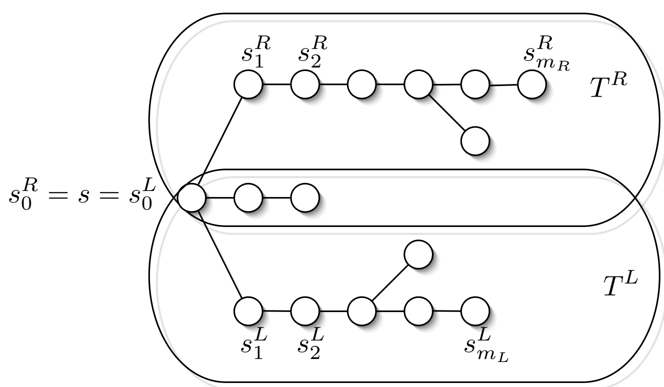

Figure 6: Notation for Lemma 4.2.

To obtain the desired bound on $|X_i|$, we bound $|X_i \cap T^R|$ and $|X_i \cap T^L|$ separately, showing $|X_i \cap T^R|, |X_i \cap T^L| \le \lfloor \frac{5d}{2} \rfloor$. The arguments are identical except for notation, so without loss of generality we argue only the former.

For $|X_i \cap T^R|$, let $j = \min\{m_R, i+d\} - \lceil \frac{d}{2} \rceil$ and $\ell = \max\{0, i-(k-d)\}$. Then $j \ge 0$ and $\ell \le i \le m_R - \lceil \frac{d}{2} \rceil$, and

$$\begin{aligned}
|X_i \cap T^R| &\le |X_i \cap T_i^R| + |X_i \cap (T^R \setminus T_i^R)| \\
&\le |X_i \cap T_j^R| + |X_i \cap (T_i^R \setminus T_j^R)| + |X_i \cap (T^R \setminus T_i^R)| \\
&\le |X_i \cap T_j^R| + |X_i \cap (T_i^R \setminus T_j^R)| + |X_i \cap (T_\ell^R \setminus T_i^R)|,
\end{aligned}$$

where the last inequality follows because any vertex $v \in (X_i \cap T^R) \setminus T_\ell^R$ is at least distance $i - \ell + 1 \geq (k - d)$ from its connection point on P, which has $\lfloor \frac{d}{2} \rfloor$ and $\lceil \frac{d}{2} \rceil$ vertices of P on either side of it, hence a forbidden subgraph. See figure 7.

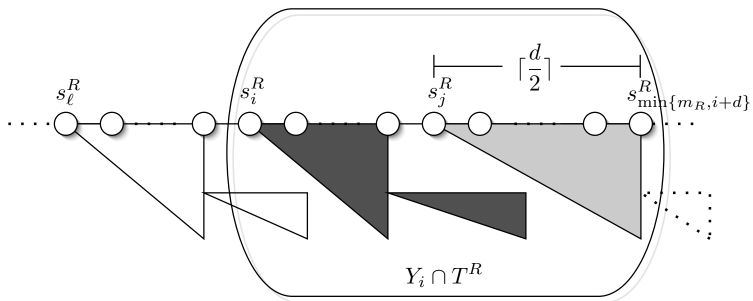

Figure 7: The above inequalities shown pictorially. The light gray triangle is $X_i \cap T_j^R$. The dark gray triangles are $X_i \cap (T_i^R \setminus T_j^R)$. The small white triangle is $X_i \cap (T_\ell^R \setminus T_i^R)$.

For $|X_i \cap T_j^R|$, notice that the vertices $s_{j-1}^R, s_{j-2}^R, \ldots, s_0^R, s_1^L, \ldots, s_{\lfloor \frac{d}{2} \rfloor - j}^L$, none in T_j^R, contain at least $\lfloor \frac{d}{2} \rfloor$ vertices within $\lfloor \frac{d}{2} \rfloor$ of v_j. Hence there can be at most $k - \lfloor \frac{d}{2} \rfloor \leq d$ vertices in T_j^R within distance $\lceil \frac{d}{2} \rceil$ of v_j, and since $T_j^R \cap X_i$ is precisely this set of vertices, we have $|T_j^R \cap X_i| \leq d$.

For $|X_i \cap (T_i^R \setminus T_j^R)|$, consider the interval of length $p = j - i \leq \lfloor \frac{d}{2} \rfloor$ from v_i to v_j. There are at least $\lfloor \frac{d}{2} \rfloor$ P vertices to its left and to its right, thus by Lemma 2.8, $|(T_i^R \setminus T_j^R) \setminus P| \leq k - d - 1$, and so $|X_i \cap (T_i^R \setminus T_j^R)| \leq k - d - 1 + p + 1 \leq k - \lceil \frac{d}{2} \rceil \leq d$.

For $|X_i \cap (T_\ell^R \setminus T_i^R)|$ consider the interval v_ℓ to v_i of length $p = i - \ell \leq \lfloor \frac{d}{2} \rfloor$. There are at least $\lfloor \frac{d}{2} \rfloor$ P vertices to its left and to its right, and hence by Lemma 2.8, $|T_\ell^R \setminus T_i^R \setminus P| \leq k - d - 1$, and so $|X_i \cap (T_\ell^R \setminus T_i^R)| \leq \lfloor \frac{d}{2} \rfloor$.

Thus $|X_i \cap T^R| \leq \lfloor \frac{5d}{2} \rfloor$, and hence $|X_i \cap H| = |X_i| \leq 5d$, as required. □

4.2.3 Computing the Coloring

Theorem 4.3 *For fixed constants d, k with $k \leq \lfloor \frac{3d}{2} \rfloor$, there is an $O(n)$ algorithm for determining whether a (d, k)-coloring of G exists and for finding one if so.*

Proof. Attempt to compute the path decomposition (T, F, X) of Lemma 4.2 as described in 4.2.2. We either obtain a path decomposition of width at most $5d$ or determine that G is not (d, k)-colorable. Assume the former. Taking

$$\hat{E} \stackrel{\text{def}}{=} \{(u, v) \mid d_G(u, v) \leq d\}$$

and $\hat{G} = (V, \hat{E})$, the decomposition (T, F, X) gives a path decomposition of $\hat{G}$ as well. The algorithms of [17, 18] can determine the $(1, k)$-colorability of $\hat{G}$ in linear time, and a $(1, k)$-coloring of $\hat{G}$ gives a (d, k)-coloring of G. □

5 Coloring on Trees

Algorithm to Find a (d, k)-coloring for a Tree T.

1. If $d = 1$ and $k \geq 2$ then we $(1, k)$-color a bipartite graph with k colors.
2. Otherwise, perform a breadth-first search on T, starting from a leaf node s to get level sets $L_0, L_1, \ldots, L_m$, where L_i consists of the set of vertices of distance i from the root s.
3. For $i = 0, 1, 2, \ldots, m$, color the vertices of L_i in some arbitrary order by assigning $v \in L_i$ the lowest color not yet assigned to any vertex within distance d of v.
4. If we need more than k colors then T is not (d, k)-colorable, otherwise return the assigned coloring.

Claim 5.1 *The algorithm returns a coloring of T if and only if T is (d, k)-colorable.*

Proof. [⇒] Consider a coloring found by the algorithm; it uses at most k colors. Moreover, no two vertices within distance d of each other are assigned the same color. Indeed, consider any two vertices u and v such that $d(u, v) \leq d$, and suppose u preceded v in the algorithm. Then when v is considered, u's color is excluded from consideration; consequently, the algorithm will not assign v the color that it used for u.

[⇐] Suppose T is (d, k)-colorable. Consider one of the vertices v, and suppose there are t vertices already colored within distance d of v. Then these t vertices along with v are within distance d of each other, and hence $t + 1 \leq k$ necessarily. Thus $t \leq k - 1$. It follows that at least one of the k colors are not excluded by this set of t vertices, and so there is a color that can be assigned to v without exceeding the allotted k colors. □

Theorem 5.2 *The (d, k)-coloring problem is polynomial-time for trees.*

Proof. The greedy algorithm runs in polynomial time and returns a (d, k)-coloring if and only if the tree is (d, k)-colorable. $\square$

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